1·11. ACCEPTANCE SAMPLING INSPECTION PLANS

In many a manufacturing process, the producer, in order to ensure that th manufactured goods are according to specifications of the customer, gets his lot checked a strategic stages. On the other hand, the customer is anxious to satisfy himself about th quality of goods he accepts. An ideal way of doing this seems to inspect each and every iter presented for acceptance, *i.e.*, to resort to 100 per cent inspection. 100% inspection should b taken recourse to under the following conditions :

(*i*) The occurrence of a defect may cause loss of life or serious casualty to personnel.

(*ii*) A defect may cause serious malfunction of equipment.

We may also wish to examine all the items of the product under the following conditions

(*i*) N, the lot size is small, and

(*ii*) The incoming quality is poor or unknown.

If testing is destructive, as for instance in the case of crackers, shells, bulbs etc, it is absolutely non-sensical to talk of 100% inspection. Even in those cases where 100% inspection is possible, it may not be desirable because (*i*) it is costly, and (*ii*) due to fatigue, impossibility of proper check, and variations in efficiencies of inspection in time, persons, and places, however careful one may be, the inspected lot is likely to contain a small percentage of defectives.

So, from practical and economic considerations sampling procedures are adopted, *i.e.*, a lot is accepted or rejected, on the basis of the samples drawn at random from the lot. It has been found that if a scientifically designed sampling inspection plan is used, it provides adequate protection to producer as well as consumer very economically.

The main objective of inspection is to control the quality of the product by critical examination at strategic points. Sampling inspection, besides keeping down the cost of production, also ensures that the quality of a lot accepted is according to the specifications of the consumer. The guidelines of a sampling procedure are :

(*a*) It should give a definite assurance against passing any unsatisfactory lot, and

(*b*) The inspection expenses should be as low as possible subject to the degree of protection afforded by (*a*) above.

Acceptance sampling plans refer to the use of sampling inspection by a purchaser to decide whether to accept or to reject a lot of given product. In statistical quality control terminology, it is also known as ***product control****. In case of acceptance sampling by attributes, the decision for accepting or rejecting a given lot is taken on the basis whether the sample items possess a particular attribute or not. In other words, acceptance sampling by attributes is an 'attribute based inspection' that merely grades the product as defective or non-defective. If the product is found defective, it is rejected and if it is found non-defective, it is accepted.*

The necessity of acceptance sampling arises from the fact that when lot of product is transferred from one firm to another, or from one division of a firm to its another division or say, from seller to the buyer, the recipient of the lot wants to be reasonably sure that the lot meets the standards already agreed upon with regard to the quality of the product. In other words, acceptance sampling prescribes a procedure, that if applied to a series of lots, yields quality assurance by involving a decision to accept or reject a lot on the basis of random samples drawn from it.

Acceptable Quality Level (AQL). This is the quality level of a good lot. It is the per-cent defective that can be considered satisfactory as a process average, and represents a level of quality which the producer wants accepted with a high probability of acceptance. In other words, if α is the producer's risk [see (1.15)], then the level of quality which results in $100(1 - \alpha)$ % acceptance of the good lots submitted for inspection is called the acceptable quality level.

A lot with relatively small fraction defective (*i.e.*, sufficiently good quality) say, p_1 that we do not wish to reject more often than a small proportion of time is sometimes referred to as a good lot. Usually,

$$P\text{ (Rejecting a lot of quality } p_1) = 0{\cdot}05$$

$$\Rightarrow \quad P_a = P\text{ (Accepting of a lot of quality } p_1) = 0{\cdot}95$$

'p_1' is known as the '*Acceptance Quality Level*' and a lot of this quality is considered as satisfactory by the consumer.

Lot Tolerance Proportion or Percentage Defective (LTPD). The *lot tolerance proportion defective*, usually denoted by p_t, is the lot quality which is considered to be bad by the consumer. The consumer is not willing to accept lots having proportion defective p_t or greater. 100 p_t is called *Lot Tolerance Percentage Defective*. In other words, this is the quality level which the consumer regards as rejectable and is usually abbreviated as *R.Q.L.* (*Rejecting Quality Level*). A lot of quality p_t stands to be accepted some arbitrary and small fraction of time, usually 10%.

Process Average Fraction Defective ($\bar{p}$). $\bar{p}$ represents the quality turned out by the manufacturing process over a long period of time. In industry, the quality of any process tends to settle down to some level which may be expected to be more or less the same everyday for a particular machine. If this level could be maintained and if the process is working free from assignable causes of variation, the inspection could often be dispensed with. But in practice, as a result of failure of machine and men, the quality for the product may suddenly deteriorate. The process average of any manufactured product is obtained by finding the percentage of defectives in the product over a fairly long time.

Consumer's Risk. Any sampling scheme would involve certain risk on the part of the consumer—in the sense that he has to accept certain percentage of undesirably bad lots, *i.e.*, lots of quality p_t or greater fraction defective. More precisely, the probability of accepting a lot with fraction defective p_t is termed as consumer's risk and is written, as P_c. Usually it is denoted by β. This is taken by Dodge and Romig as 10% or 0·10.

$$\text{Consumer's risk} = P_c = P \text{ [accepting a lot of quality } p_t] = \beta \quad \ldots(1\cdot14)$$

Producer's Risk. The producer has also to face the situation that some good lots will be rejected. He might demand adequate protection against such contingencies happening too frequently just as the consumer can claim reasonable protection against accepting too many bad lots. The probability of rejecting a lot with 100 $\bar{p}$ as the process average percentage defective is called the producer's risk P_p and is usually denoted by α. Thus

$$\text{Producer's risk} = P_p = P \text{ (of rejecting a lot of quality } \bar{p}) = \alpha \quad \ldots(1\cdot15)$$

Rectifying Inspection Plans. In the following sections we shall discuss lot by lot sampling plans in which a specified quality objective is attained through corrective inspection of rejected lots. The inspection of the rejected lots and replacing the defective pieces found in the rejected lots by the good ones, eliminates the number of defectives in the lot to a great extent, thus improving the lot quality. These plans are called '*Rectifying Inspection Plans*' and were first introduced by Harold F. Dodge and Harry G. Romig of the Bell Telephone Laboratories before World War II. These plans enable the manufacturer to have an idea about the average quality of the product that is likely to *result* at a given stage of manufacture through the combination of production, sampling inspection and rectification of rejected lots.

Most of the rectifying inspection plans for lot by lot sampling call for 100% inspection of the rejected lots and replacing the defective pieces found by good ones. The two important points related to rectifying inspection plans are :

(*i*) The average quality of the product after sampling and 100% inspection of rejected lots, called Average Outgoing Quality (*AOQ*); and

(*ii*) The average amount of inspection required for the rectifying inspection plan, called Average Total Inspection (*ATI*).

***Average Outgoing Quality Limit* (*AOQL*).** Sometimes the consumer is guaranteed a certain quality level after inspection—regardless of what quality level is being maintained by the producer. Let the producer's fraction defective, *i.e.*, lot quality before inspection be 'p'. This is termed as '*incoming quality*'. The fraction defective of the lot after inspection is known as '*outgoing quality*' of the lot. The expected fraction defective remaining in the lot after the application of the sampling inspection plans is termed as Average Outgoing Quality $(AOQ)\,\bar{p}$. Obviously, it is a function of the incoming quality 'p'.

Remark. For rectifying inspection single sampling plan (see § 1·12) calling for 100% inspection of the rejected lots, the AOQ values are given by the formula :

$$\bar{p} = AOQ = \frac{p(N-n)\,P_a}{N} \qquad \ldots(1\cdot16)$$

where N is lot size, n is sample size and P_a is the probability of acceptance of the lot.

Formula (1·16) assumes that all defectives found are repaired or replaced by good pieces.

Since we look for defective pieces in the uninspected portion of accepted lots (involving $N-n$ items) and since p is the probability of finding a defective, there will on the average be $p(N-n)$ defective items. Since P_a is the probability of acceptance of the lot, the sampling plan will on the average turn out lots that contain $p.P_a\ (N-n)$ defective items. Consequently, on dividing by N, we get AOQ as a fraction defective given by (1·16).

If n is small compared with N, then a good approximation of the outgoing quality is given by :

$$\bar{p} = \text{AOQ} = p\,P_a \qquad \ldots(1\cdot16a)$$

If the defective pieces found are not repaired or replaced, then the formula must be modified to

$$\text{AOQ} = \frac{p(N-n)\,P_a}{N-np-p(1-P_a)\,(N-n)} = \frac{p(N-n)\,P_a}{N-p[nP_a+N(1-P_a)]} \qquad \ldots(1\cdot16b)$$

This formula is not generally used and if p is small, there is not much difference between (1·16a) and (1·16b).

In general, if p is the incoming quality and a rectifying inspection plan calling for 100% inspection of the rejected lots is used, then the AOQ of the lot will be given by :

$$\text{AOQ} = p\,P_a(p) + 0\cdot[1-P_a(p)] = pP_a(p) \qquad \ldots(1\cdot16c)$$

because (i) $P_a\,(p)$ is the probability of accepting the lot of quality 'p' and when the lot is accepted on the basis of the inspection plan, the outgoing quality of the lot will be approximately same as the incoming lot quality 'p'; and

(ii) $1-P_a(p)$ is the probability of rejection of the lot and when the lot is rejected after sampling inspection and is subjected to 100% screening and rectification, the AOQ is zero.

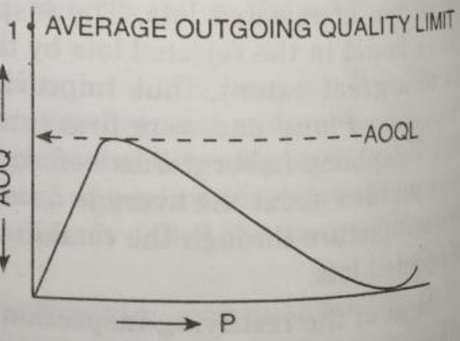

Fig. 1·17

For a given sampling plan, the value of AOQ can be plotted for different values of p to obtain the AOQ curve as given in *Fig. 1·17*:

From (1·16c), we find that if $p = 0$, *i.e.*, the lot is 100% O.K. then AOQ = 0 and if $p = 1$ *i.e.*, lot is 100% defective then $P_a(p) = 0$ and so AOQ = 0. For other values of p lying between 0 and 1, the AOQ will be positive and will have a maximum value for some value of the incoming quality p. The maximum value of $\bar{p}$ subject to variation in p is called the Average Outgoing Quality Limit ($\bar{p}_L$). If p_M is the value of p which maximises $\bar{p}$ in (1·16), then

$$\bar{p}_L = \text{A.O.Q.L.} = \frac{p_M \cdot P_a(N-n)}{N} \qquad \ldots(1\cdot16d)$$

where it should be remembered that P_a is to be computed for $p = p_M$. Re-writing (1·16d), we have

$$\text{AOQL} = \frac{y}{n}\left(1 - \frac{n}{N}\right) \qquad \ldots(1\cdot16e)$$

where $y = np_M$. P_a, has been tabulated by Dodge and Romig for various values of n (sample size) and c (acceptance number of sampling plan, *i.e.*, maximum allowable number of defectives in the sample). The AOQL measures the long-term protection given by the plan to the user in the worst situation.

It may be pointed out that AOQL *curve* ($\bar{p}_L$ plotted against p) will reach a maximum value and then recede, since the poorer the quality of the incoming product (*i.e.*, larger the value of p) the fewer lots will be accepted and more will be inspected 100% and made acceptable.

OC Curve. Operating characteristic (OC) curve of a sampling plan is a graphic representation of the relationship between the probability of acceptance $P_a(p)$ or generally denoted by $L(p)$, for variations in the incoming lot quality 'p' (fraction defective in the lot). For five general points on the OC curve, see § 1·12·4.

Average Sample Number (ASN) and Average Amount of Total Inspection. The *average sample number (ASN)* is the expected value of the sample size required for coming to a decision about the acceptance or rejection of the lot in an acceptance-rejection sampling plan. Obviously it is a function of the incoming lot quality p. On the other hand, the expected number of items inspected per lot to arrive at a decision in an acceptance-rectification sampling inspection plan calling for 100% inspection of the rejected lots is called average amount of total inspection (ATI). Obviously ATI is also a function of the lot quality p.

We observe that

ATI = ASN + (Average size of inspection of the remainder in the rejected lots) ...(1·17)

Thus, if the *lot is accepted on the basis of the sampling inspection plan then ATI = ASN, otherwise ATI > ASN*. In other words ASN gives the average number of units inspected per accepted lot.

For example, if a single sampling acceptance - rejection plan is used, the number of items inspected from each lot will be the corresponding sample size n, *i.e.*,

$$\text{ASN} = n, \qquad \ldots(1\cdot17a)$$

and this will be true, independently of the quality of the submitted lots.

However, for an acceptance-rectification single sampling plan calling for 100% inspection of the rejected lots, additional $(N-n)$ items will have to be inspected for each rejected lot,

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where N is the lot size. Thus, in this case, the number of items inspected per lot varies from lot to lot and is equal to n if the lot is accepted and equal to N if the lot is rejected on the basis of the sampling inspection plan. Hence the average amount of total inspection is a function of the lot quality 'p' and is given by :

$$\text{ATI} = nL(p) + N[1 - L(p)] \quad \ldots(1\cdot17b)$$

where $L(p) = P_a(p)$ is the probability of acceptance of the lot of quality p on the basis of the sampling inspection. Rewriting (1·17b), we get

$$\begin{aligned}\text{ATI} &= nL(p) + (N - n + n)\,[1 - L(p)]\\ &= nL(p) + (N - n)\,[1 - L(p)] + n[1 - L(p)]\\ &= n + (N - n)\,[1 - L(p)] \quad \ldots(1\cdot17c)\end{aligned}$$

Remarks **1.** In single sampling inspection plan, the common practice is to inspect the entire sample even though the decision to accept or reject the lot is reached before the entire sample is inspected. Hence ASN in a single sampling plan is n. Similarly in a double sampling plan, the entire first sample is always inspected.

2. The actual sample size cannot be fractional but the expected sample size may be obtained to the nearest decimal required.

3. The ASN and ATI plotted against the lot quality 'p' give the ASN curve and ATI curve respectively. These are useful in comparing the efficiency and costs of the sampling inspection plans. In some situations like destructive testing or variables inspection, the rectification of the rejected lot is not feasible or practicable and the ATI curves are not used. In such situations ASN curve is used.

1·12. SAMPLING INSPECTION PLANS FOR ATTRIBUTES

The commonly used sampling inspection plans for attributes and count of defects are :

(*i*) Single sampling plan, (*ii*) Double sampling plan, and , (*iii*) Sequential sampling plan.

The requirements (a) and (b) in § 1·11 on page 1·45 will be satisfied provided p_t, p and P_t are low. Using these principles, H.E. Dodge and H.G. Romig have developed a number of sampling plans which we shall discuss below. These plans enable us to judge the average quality of the product at a given stage of manufacturing process through the combination of production, sampling inspection and rectification of rejected lots. Dodge and Romig average quality protection plans are essentially based upon the AOQL.

1·12·1. Single Sampling Plan. If the decision about accepting or rejecting a lot is taken on the basis of one sample only, the acceptance plan is described as single sampling plan. It is completely specified by three numbers N, n and c, where :

N is the lot size,

n is the sample size, and

c is the acceptance number, *i.e.*, maximum allowable number of defectives in the sample.

The single sampling plan may be described as follows :

1. Select a random sample of size n from a lot of size N.

2. Inspect all the articles included in the sample. Let d be the number of defectives in the sample.

3. If $d \le c$, accept the lot, replacing defective pieces found in the sample by non-defective (standard) ones.

FLOW-CHART OF SINGLE SAMPLING RETIFICATION PLAN

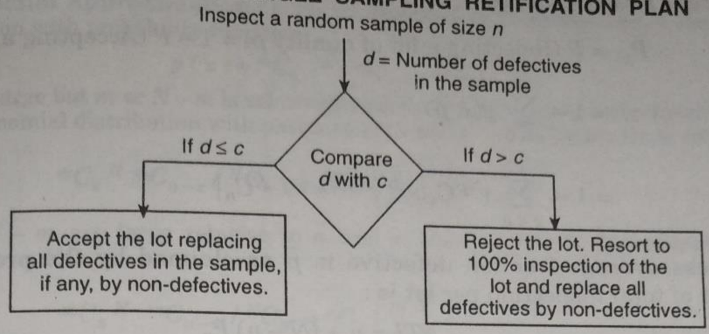

Fig. 1·18

4. If $d > c$, reject the lot. In this case we inspect the entire lot and replace all the defective pieces by standard ones.

The flow-chart [*Fig. 1·18*] elegantly displays the single sampling Rectification plan.

Single sampling plan is very simple to understand, design and carry out. The basic problem in administering a single sampling plan is the choice of n (*sample size*) and c (*acceptance number*) which have to be determined in advance. The most economical single sampling inspection plan is obtained on minimising the average total inspection by providing adequate protection to consumer and producer. Dodge and Romig have prepared extensive tables for minimising values of n and c for consumer's risk $\beta = 0{\cdot}10$ and for different values of p (the process average fraction defective).

Remark. Obviously in such a plan, the chance of cent-per-cent inspection increases as the percentage of defectives in the lot increases. Thus, the amount of inspection automatically increases as the lot quality deteriorates.

Determination of n and c. The lot size N is invariably known. Thus the two unknown quantities that need to be determined in the sampling plan are n and c.

In a lot of incoming quality p, the number of defective pieces is Np and non-defective pieces is $N - Np = N(1 - p)$. The probability of getting exactly x defectives in a sample of size n from this lot is given by (Hyper-geometric distribution).

$$g(x, p) = [{}^{Np}C_x \times {}^{N-Np}C_{n-x} / {}^{N}C_n]$$

Probability of accepting a lot of quality p is

$$P_a(p) = \sum_{x=0}^{c} g(x, p) = \sum_{x=0}^{c} [{}^{Np}C_x \times {}^{N-Np}C_{n-x} / {}^{N}C_n] \qquad \ldots(1{\cdot}18)$$

Hence the *consumer's risk* is given by

$$P_c = P\,\{\text{Accepting a lot of quality } p_t\}$$

$$= \sum_{x=0}^{c} g(x, p_t) = \sum_{x=0}^{c} {}^{Np_t}C_x \times {}^{N-Np_t}C_{n-x} / {}^{N}C_n \qquad \ldots(1{\cdot}18a)$$

To protect himself against poor quality, the consumer usually demands a small value of P for given p.

The *producer's risk* is given by :

$$P_p = P \text{ (Rejecting a lot of quality } \bar{p}) = 1 - P \text{ (Accepting a lot of quality } \bar{p})$$

$$= 1 - \sum_{x=0}^{c} g(x, \bar{p})$$

$$= 1 - \sum_{x=0}^{c} [{}^{N\bar{p}}C_x \times {}^{N-N\bar{p}}C_{n-x} \,/\, {}^{N}C_n] \quad \text{...(1·18}b)$$

If the process average fraction defective is $\bar{p}$ as claimed by the producer, then the average amount of total inspection per lot is :

$$ATI = n + (N - n)\, P_p \quad \text{...(1·18}c)$$

since n items have to be inspected in each case and the remaining $(N - n)$ items will be inspected only if $d > c$, *i.e.*, if the lot is rejected when the lot quality is $\bar{p}$. and the probability for this is P_p.

The computation of hyper-geometric probabilities in (1·18a) and (1·18b) is extremely difficult and, in practice, the binomial approximation to hyper-geometric distribution [see (*) and (**) in the remark below] is used.

Using (*), a convenient and practical substitute for (1·18a) becomes :

$$P_c = \sum_{x=0}^{c} \frac{(Np_t)\,!}{x\,!\,(Np_t - x)\,!} \left(\frac{n}{N}\right)^x \left(1 - \frac{n}{N}\right)^{Np_t - x} \quad \text{...(1·18}d)$$

and using (**), we get from (1·18b)

$$P_p = 1 - \sum_{x=0}^{c} \frac{n\,!}{x\,!(n-x)\,!} (\bar{p}\,)^x (1 - \bar{p})^{n-x} \quad \text{...(1·18}e)$$

In most of the practical problems, $\bar{p}$ is likely to be less than 0·10 and n is likely to be sufficiently large to warrant the use of Poisson approximation to binomial distribution. Thus, (1·18e) can further be approximated by :

$$P_p = 1 - \sum_{x=0}^{c} \left[\frac{(n\,\bar{p})^x \exp(-n\,\bar{p}\,)}{x\,!}\right] \quad \text{...(1·18}f)$$

and consequently

$$ATI = n + (N - n) \left[1 - \sum_{x=0}^{c} \left\{\frac{exp\,(-\,n\bar{p}_1)\,(n\bar{p})^x}{x\,!}\right\}\right] \quad \text{...(1·18}g)$$

Consumer's requirement fixes the values of P_c and p_t. N is always fixed. For given values of P_c and p_t the equation (1·18a) which involves two unknowns n and c is satisfied by a large number of pairs of n and c. To safeguard producer's interest also, out of these possible pairs one involving the minimum amount of inspection as given in (1·18c) is chosen. Though theoretical computations are quite cumbersome and time consuming, Dodge and Romig, by applying numerical methods of solution of equations, have prepared extensive tables for minimising values of n and c for $P_c = 0{\cdot}10$ and different values of $\bar{p}$.

Remark. **Binomial Approximation to Hyper-geometric Distribution.** Consider the hyper-geometric distribution with probability function :

$$p(x) = {}^mC_x \cdot {}^{N-m}C_{n-x} / {}^NC_n \qquad ...(*)$$

If N and n are large but m or $N - m$ is relatively small, then hyper-geometric distribution (*) can be approximated by binomial distribution with parameters m and $p = n/N$. Under these conditions, we get

$${}^mC_x\, {}^{N-m}C_{n-x} / {}^NC_n \to {}^mC_x \left(\frac{n}{N}\right)^x \left(1 - \frac{n}{N}\right)^{m-x} \qquad ...(1)$$

If N, m and $N - m$ are large relative to n and x, then (*i*) can be approximated by binomial distribution $B(n, p = m/N)$.

Thus
$${}^mC_x\, {}^{N-m}C_{n-x} / {}^NC_n \to {}^nC_x \left(\frac{m}{N}\right)^x \left(1 - \frac{m}{N}\right)^{n-x} \qquad ...(**)$$

AOQL. If p is the incoming quality, there will be no defectives left in a lot of size N if the sample contains more than c defectives, *i.e.,* if $x > c$. On the other hand, if $x \le c$, the number of defectives in a lot of size N is $Np - x$. Thus, the mean value of the number of defectives after sampling inspection is given by :

$$m = \sum_{x=0}^{c} (Np - x)\, g(x, p) + \sum_{x=c+1}^{N} 0 \cdot g(x, p)$$

$$= \sum_{x=0}^{c} (Np - x)\, {}^{Np}C_x\, {}^{N-Np}C_{n-x} / {}^NC_n$$

The mean values of the fraction defective after inspection, *i.e.,* AOQ will be

$$\text{AOQ} = \tilde{p} = \frac{m}{N} = \sum_{x=0}^{c} \left(p - \frac{x}{N}\right) {}^{Np}C_x\, {}^{N-Np}C_{n-x} / {}^NC_n \qquad ...(1\cdot19)$$

Subject to variation in p, $\tilde{p}$ has a maximum value, say, $\tilde{p}_L$ which is termed as A.O.Q.L. A.O.Q.L. takes care of consumer's interest.

Given N and $\tilde{p}_L$ (AOQL), the equation (1·18*a*) contains two unknowns n and c. It is possible to select several pairs of values of n and c that will give $\tilde{p}$ as defined in (1·19) having approximately the same value of $\tilde{p}_L$. As a safeguard to producer's interests, we select the pair which minimises ATI as defined in (1·18*c*). Dodge and Romig have prepared extensive AOQL tables for minimising values of n and c.

OC Curve. The OC curve for the incoming quality 'p' is given by :

$$P_a(p) = L(p) = \sum_{x=0}^{c} g(x, p)$$

$$= \sum_{x=0}^{c} \left[{}^{Np}C_x \times {}^{N-Np}C_{n-x} / {}^NC_n \right] \qquad ...(1\cdot20)$$

When $p < 0{\cdot}10$, a good approximation to (1·20) is given by the first $(c + 1)$ terms of the binomial expansion $\left[\left(1 - \frac{n}{N}\right) + \frac{n}{N}\right]^{Np}$

i.e.,
$$L(p) \simeq \sum_{x=0}^{c} {}^{Np}C_x \left(\frac{n}{N}\right)^x \left(1 - \frac{n}{N}\right)^{Np-x} \qquad ...(1\cdot20a)$$

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When $p < 0{\cdot}10$ and also $\frac{n}{N} < 0{\cdot}10$, a good approximation to (1·20) is given by the Poisson distribution, *viz.*,

$$L(p) = \sum_{x=0}^{c} \left[e^{-np} (np)^x / x\,! \right]. \qquad \ldots (1{\cdot}20b)$$

Example 1·16. *From a lot consisting of 2,200 items, a sample of size 225 is taken. If it contains 14 or less defectives, the lot is accepted otherwise rejected. Plot the OC, ATI and AOQ curves. Also obtain the value of AOQL.*

Solution. Here we have a single sampling plan characterised by :

$$N = 2200\ ,\quad n = 225\ ,\qquad c = 14.$$

Using Poisson approximation to hyper-geometric distribution, the OC curve, *i.e.*, the probability of acceptance $L(p)$ or P_a is given by (*c.f.* 1·20*b*)

$$P_a = \sum_{x=0}^{c} \frac{e^{-np}\,(np)^x}{x\,!} = \sum_{x=0}^{c} \frac{e^{-\lambda}\,\lambda^x}{x\,!} \qquad \ldots(*)$$

where $\lambda = np$ and p is the submitted lot quality.

The probabilities in (*) for different values of λ and c can be obtained from Table 7 in Biometrika Tables for Statisticians. We compute P_a in (*) for different values of p from 0 to 1.

The average total amount that is inspected under the above sampling rectification inspection plan is given by [*c.f.* {1·18 (*g*)}]

$$\text{ATI} = n + (N - n)\,(1 - P_a) \qquad \ldots(**)$$

Also from (1·16*a*), $\quad \text{AOQ} = p.P_a \ldots(***)$

COMPUTATION OF ATI AND AOQ VALUES

p	$np = \lambda$	$\sum_{x=c}^{\infty} \frac{e^{-\lambda}\lambda^x}{x\,!}$	P_a	$A.O.Q. = p.\,P_a$	$(N-n) \times (1-P_a)$	$ATI = n + (N-n) \times (1-P_a)$
(1)	(2)	(3)	(4) = (1) − (3)	(5) = (1) × (4)	(6)	(7) = n + (6)
·01	2·25	·0	1·00000	·01000	0	225·00
·02	4·50	·000074	·999926	·019998	·146150	225·146
·03	6·75	·004444	·995566	·029866	8·757150	233·757
·04	9·00	·041466	·958534	·038341	81·895350	306·895
·05	11·25	·168660	·831340	·041567	333·1035	558·103
·06	13·50	·376729	·623271	·37396	744·039775	969·039
·08	18·00	·791623	·208077	·16646	1564·047925	1789·048
·10	22·50	·968926	·031074	·003107	1913·62885	2138·629
·30	67·50	1·00000	0	0	1975·00	2200·000
·80	180·0	1·00000	0	0	1975·00	2200·000
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The OC Curve, the ATI curve and the AOQ curve of the above sampling plan are drawn in Fig. 1·19, Fig. 1·20 and Fig. 1·21 respectively.

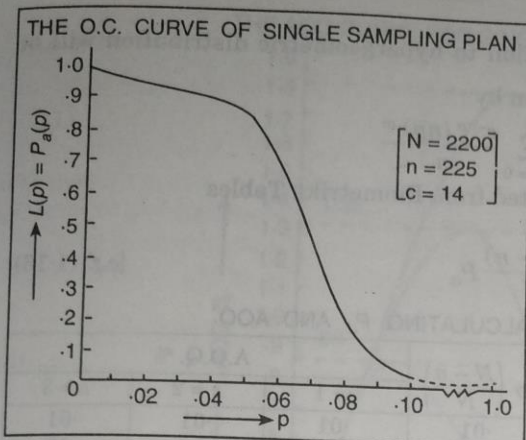

Fig. 1·19

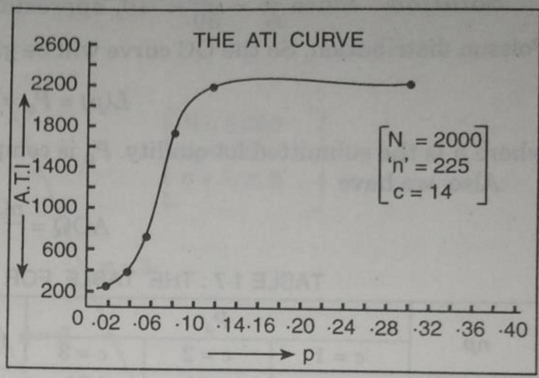

Fig. 1·20

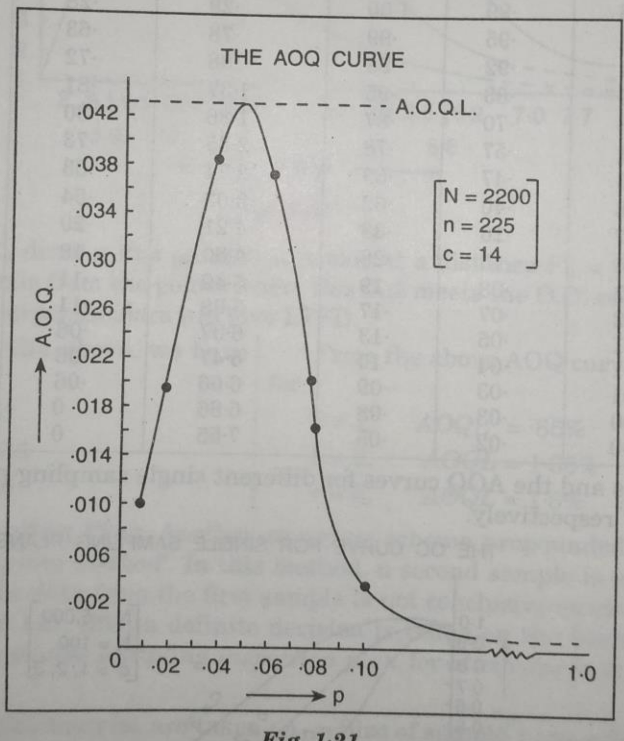

Fig. 1·21

Example 1·17. *(a) Plot the operating characteristic curves for Single Sampling Plan where $N = 5{,}000$; $n = 100$; $c = 1, 2, 3$. Assuming $P_c = \cdot 10$, determine the lot tolerance fraction defective.*

(b) For the above data, plot the average outgoing quality (AOQ) curve and determine AOQL.

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Solution. Since $\frac{n}{N} = \frac{1}{50} < \cdot 10$, approximation to hypergeometric distribution will be Poisson distribution. So the OC curve will be given by :

$$L(p) = P_a = \sum_{m=0}^{c} \frac{e^{-np}\,(np)^m}{m\,!}$$

where p is the submitted lot quality. P_a is computed from Biometrika Tables

Also, we have

$$AOQ = \frac{p(N-n)}{N} P_a \qquad \text{[c.f. (1·16)]}$$

TABLE 1·7 : THE TABLE FOR CALCULATING P_a AND AOQ

np	P_a			$100p\left(\frac{N-n}{N}\right)$	A.O.Q. %		
	$c = 1$	$c = 2$	$c = 3$		$c = 1$	$c = 2$	$c = 3$
·01	1·00	1·00	1·00	·01	·01	·01	·01
·20	·98	·99	1·00	·19	·19	·18	·19
·25	·97	·99	1·00	·24	·23	·24	·29
·3	·96	·99	1·00	·29	·28	·29	·29
·8	·81	·95	·99	·78	·63	·74	·77
1·0	·74	·92	·98	·98	·72	·90	·96
1·4	·59	·83	·95	1·37	·81	1·14	1·30
1·9	·43	·70	·87	1·86	·80	1·30	1·62
2·4	·31	·57	·78	2·35	·73	1·34	1·83
2·8	·23	·47	·69	2·74	·63	1·29	1·89
3·1	·18	·40	·62	3·03	·54	1·29	1·88
4·3	·07	·20	·38	4·21	·29	·84	1·60
4·9	·04	·13	·28	4·80	·19	·62	1·34
5·6	·02	·08	·19	5·49	·11	·44	1·04
5·8	·02	·07	·17	5·68	·11	·40	·96
6·2	·01	·05	·13	6·07	·06	·30	·79
6·6	·01	·04	·10	6·47	·06	·26	·64
6·8	·01	·03	·09	6·66	·06	·20	·60
7·0	·00	·03	·08	6·86	0	·20	·54
7·7	·00	·02	·05	7·55	0	·15	·38

The OC curves and the AOQ curves for different single sampling plans are drawn in *Fig. 1·22* and *Fig. 1·23* respectively.

THE OC CURVE FOR SINGLE SAMPLING PLANS

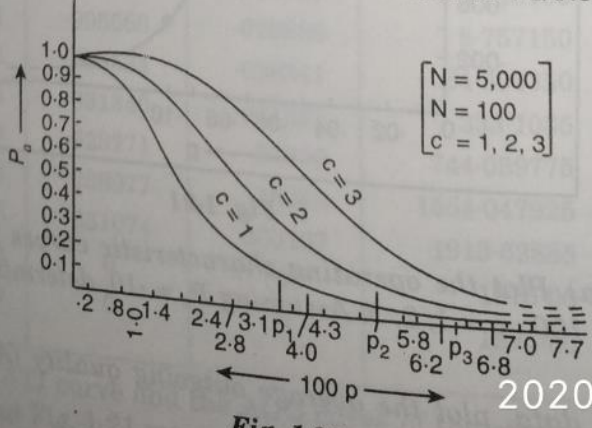

Fig. 1·22

THE AOQ CURVE FOR SINGLE SAMPLING PLANS

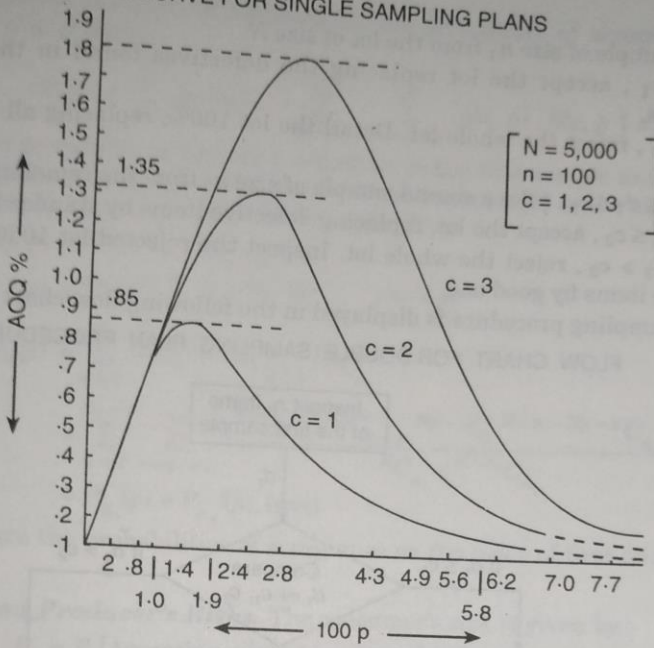

Fig. 1·23

To determine LTFD., draw a line parallel to p-axis at a distance $P_a = 0{\cdot}10$. Then draw perpendiculars to the p-axis from the points where this line meets the O.C. curves for $c = 1, 2, 3$. The abscissa of these perpendiculars will give LTFD.

For $P_c = 0{\cdot}10$, from the above, we have for

$c = 1$, $LTFD = 0{\cdot}037$
$c = 2$, $LTFD = 0{\cdot}0515$
$c = 3$, $LTFD = 0{\cdot}066$

From the above AOQ curves, we see that for

$c = 1$, $AOQL = {\cdot}85\%$
$c = 2$, $AOQL = 1{\cdot}35\%$
$c = 3$, $AOQL = 1{\cdot}8\%$

1·12·2. Double Sampling Plan. Another sampling scheme propounded by Dodge and Romig is the '*second sampling method*'. In this method, a second sample is permitted if the first sample fails *i.e.*, if the data from the first sample is not conclusive on either side (about accepting or rejecting the lot), then a definite decision is taken on the basis of the second sample. Such a *rectifying double sampling inspection plan* for *attributes* is briefly described below :

N = Lot size from which samples are taken ; n_1 = Size of sample 1 ; n_2 = Size of sample 2

c_1 = Acceptance number for first sample, *i.e.*, maximum permissible number of defectives in first sample if lot is to be accepted without taking another sample.

c_2 = Acceptance number for samples 1 and 2 combined, *i.e.*, maximum permissible number of defectives in combined samples if lot is to be accepted.

d_1 = Number of defectives in sample 1;

d_2 = Number of defectives in sample 2.

Procedure :

(*i*) Take a sample of size n_1 from the lot of size N.

(*ii*) If $d_1 \le c_1$, accept the lot replacing the defectives found in the sample by non-defectives.

(*iii*) If $d_1 > c_2$, reject the whole lot. Detail the lot 100%, replacing all bad items by good ones.

(*iv*) If $c_1 + 1 \le d_1 \le c_2$, take a second sample of size n_2 from the remaining lot.

(*v*) If $d_1 + d_2 \le c_2$, accept the lot, replacing defective items by standard ones.

(*vi*) If $d_1 + d_2 > c_2$, reject the whole lot. Inspect the rejected lot 100%, replacing all the defective items by good one.

The above sampling procedure is displayed in the following flow chart [*Fig. 1·24*] :

FLOW CHART FOR DOUBLE SAMPLING PLAN PROCEDURE

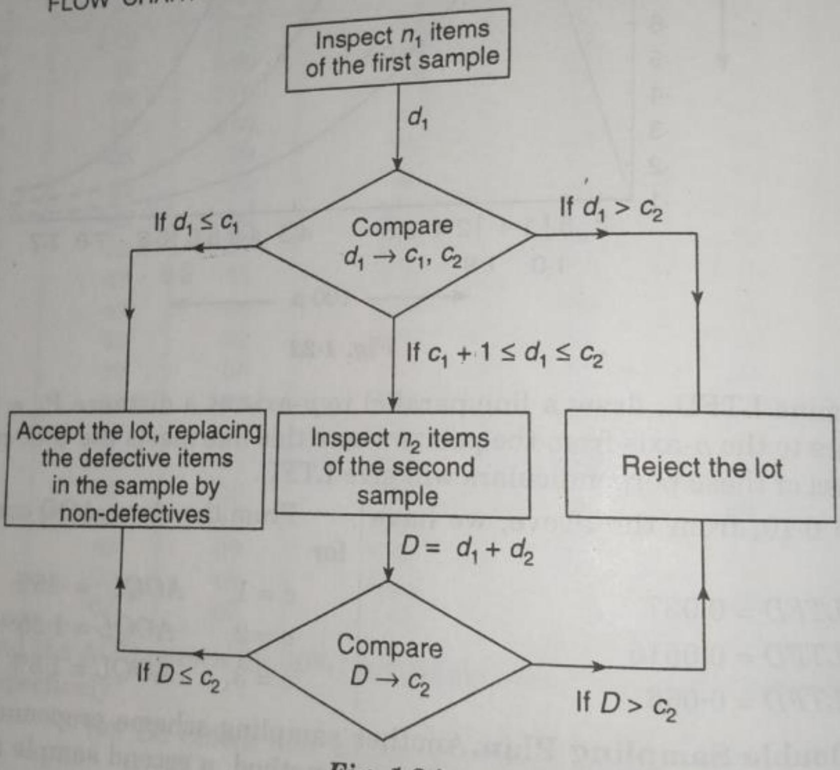

Fig. 1·24.

Dodge and Romig obtained the most economical double sampling plans after providing adequate protection to producer and consumer such that :

(*i*) Average total inspection is minimum, and

(*ii*) The probability of acceptance on the basis of first sample is same as the probability of acceptance on the basis of second sample.

O.C. Curve of Double Sampling Plan. The lot will be accepted under the following mutually exclusive ways :

(*i*) $0 \le d_1 \le c_1$; (*ii*) $d_1 = c_1 + 1, d_2 \le c_2 - c_1 - 1$; (*iii*) $d_1 = c_1 + 2$; $d_2 \le c_2 - c_1 - 2$

⋮ ⋮

$(\cdot)\ d_1 = c_2, d_2 = 0$

Hence, by addition theorem of probability, the probability of acceptance for a lot of incoming quality 'p' is given by :

$$P_a(p) = \sum_{x=0}^{c_1} g(x, p) + \sum_{y=0}^{c_2 - x} \sum_{x=c_1+1}^{c_2} g(x, p) \,.\, h(y, p \mid x)$$

where $g(x, p)$ is the probability of finding x defectives in the first sample and $h(y, p \mid x)$ is the conditional probability of finding y defectives in the second sample under the condition that x defectives have already appeared in the first sample. Thus

$$g(x, p) = {}^{Np}C_x \; {}^{N-NP}C_{n_1 - x} \,/\, {}^{N}C_{n_1}$$

$$h(y, p \mid x) = {}^{Np-x}C_y \; {}^{N-n_1-(Np-x)}C_{n_2-y} \,/\, {}^{N-n_1}C_{n_2}$$

$$\therefore \quad P_a(p) = \sum_{x=0}^{c_1} {}^{Np}C_x \; {}^{N-Np}C_{n_1-x} \,/\, {}^{N}C_{n_1}$$

$$+ \sum_{y=0}^{c_2-x} \sum_{x=c_1+1}^{c_2} \frac{{}^{Np}C_x \; {}^{N-Np}C_{n_1-x} \; {}^{Np-x}C_y \; {}^{N-n_1-Np+x}C_{n_2-y}}{{}^{N}C_{n_1} \times {}^{N-n_1}C_{n_2}} \qquad \ldots(1\cdot21)$$

$$= P_{a_1}(p) + P_{a_2}(p), \text{ (say)}$$

where P_{a_1} and P_{a_2} are the probabilities of acceptance on the basis of first and second samples respectively.

Consumer's and Producer's Risks. The consumer's risk is given by :

$$P_c = P \text{ [Accepting a lot of quality } p_t] = P_a(p_t) \qquad \ldots(1\cdot21a)$$

and producer's risk is given by

$$P_p = 1 - P \text{ [Accepting a lot of quality } \bar{p}] = 1 - P_a(\bar{p}) \qquad \ldots(1\cdot21b)$$

Hence replacing 'p' by p_t and $\bar{p}$ in (1·21), and substituting in (1·21a) and (1·21b), we get consumer's and producer's risk respectively.

ASN and ATI of Double Sampling Plan. In an acceptance-rejection double sampling plan, the number of items inspected for a lot is either n_1, (when the lot is accepted or rejected on the basis of the first sample or $(n_1 + n_2)$ when a second sample of size n_2 is drawn. Thus the expected sample size for a decision is given by :

$$ASN = n_1 P_1 + (n_1 + n_2)(1 - P_1)$$

$$= n_1 + n_2 (1 - P_1)$$

where P_1 is the probability of a decision (acceptance or rejection of the lot) on the basis of the first sample.

However, in a double sampling acceptance-rectification scheme in which rejected lots are inspected 100%, the average total inspection (ATI) per lot is given by :

$$ATI = n_1 P_{a_1} + (n_1 + n_2) P_{a_2} + N(1 - P_a), \qquad \ldots(1\cdot22)$$

since (*i*) only n_1 items will be inspected if the lot is accepted on the basis of the first sample and its probability is $P_{a_1}(p)$,

(*ii*) $(n_1 + n_2)$ items will be inspected if the lot is accepted on the basis of the second sample and its probability is $P_{a_2}(p)$, and

(*iii*) the entire lot of N items will be inspected if the lot is rejected and the probability of this is $1 - P_a(p)$.

$$P_a = P_{a_1} + P_{a_2} \Rightarrow P_{a_2} = P_a - P_{a_1},$$

Since we get from (1·22)

$$ATI = n_1 P_{a_1} + (n_1 + n_2)(P_a - P_{a_1}) + N(1 - P_a)$$
$$= n_1 P_{a_1} + (n_1 + n_2)[(1 - P_{a_1}) - (1 - P_a)] + N(1 - P_a)$$
$$= n_1 P_{a_1} + (n_1 + n_2)(1 - P_{a_1}) + (N - n_1 - n_2)(1 - P_a)$$
$$= n_1 + n_2(1 - P_{a_1}) + (N - n_1 - n_2)(1 - P_a) \quad \ldots(1\cdot22a)$$

Remark. In Dodge and Romig tables, n_2 has no fixed relation to n_1 but is determined so that ATI is minimum and so that the probability of acceptance on the basis of first sample is approximately the same as the probability of acceptance on the basis of second sample.

1·12·3. Single Sampling *vs.* Double Sampling Plans

1. Single sampling plans are simple, easy to design and administer, and since each sample can be plotted on a control chart, maximum information concerning the lot can be obtained.

2. A very important advantage of double sampling over single sampling seems to be psychological. To a layman, it seems unfair to reject a lot on the basis of one sample alone and appears more convincing to say that the lot was rejected after inspecting two samples. Moreover, in double sampling no lot can be rejected without finding at least two defectives in the sample taken from it—thus the border-line lots (lots of marginal quality) always get a second chance of being accepted.

3. In the conditions under which the sampling schemes are generally operated it has been found that the double sampling scheme involves on the average less amount of inspection than the single sampling scheme for the same quality assurance. Under the double sampling scheme the good quality lot will generally be accepted and bad lots will usually be rejected on the basis of the first sample. Thus, in all the cases, where a decision to accept or reject is taken on the basis of the first sample, there is a considerable saving in the amount of inspection than required by a comparable (*w.r.t.* OC curve) single sampling plan. Moreover, whenever a second sample is taken it may be possible to reject the lot without completely inspecting the entire second sample. Usually double sampling requires 25% to 33% less inspection on the average, than single sampling.

The general reduction in the amount of inspection afforded by double sampling is one of its strongest advantages. This does not necessarily mean, however, that a double sampling scheme could be less costlier than the single sampling scheme. The double sampling schemes being more complicated and the necessity of inspecting second sample being unpredictable, the unit cost of inspection for a double sampling procedure may be higher than that for single sampling procedure.

4. The operating characteristic curves of double sampling scheme are generally steeper than those of corresponding single sampling procedure *i.e.,* the discriminatory power of double sampling procedures is a bit higher than that of single sampling procedures.

1·12·4. Sequential Sampling Plan. We know that one of the advantages of double sampling over single sampling is psychological in giving the lot second chance for acceptance. Moreover, except for lots of marginal quality the average amount of inspection in double sampling is less for the same protection. It is, therefore, natural to suggest triple, quadruple or in general multiple sampling as a way to reduce the amount of inspection still further. Unfortunately such plans become very complex both to construct and to administer and the small gain in sampling reduction is inefficient to warrant them unless one goes all the way to *sequential sampling*.

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The ultimate in multiple sampling is sequential sampling which provides for infinite number of stages for arriving at a decision. In sequential sampling, sample items are examined one at a time and after each item inspected one of three decisions, *viz*., to accept the lot, to reject the lot or to continue sampling is taken. This scheme provides for a minimum amount of inspection.

Sequential schemes are considered to require most care and supervision in operation. Where the inspection or testing costs per article are high and sampling destructive, utmost economy in the number of articles inspected is important and often outweighs administrative convenience.

Sequential Probability Ratio Test **(*S.P.R.T.*)**. A sampling plan satisfying the condition that the probability of rejecting the lot does not exceed α whenever $p \le p_0$ and the probability of accepting lots does not exceed β whenever $p \ge p_1$ is given by the *sequential probability ratio test* (SPRT), pioneered by Dr. Abraham Wald, for testing the hypothesis H_0 : $p = p_0$ against the hypothesis $H_1 : p = p_1$.

Here if we take AQL = p_0 ; LTPD = $100p_1$ or lot tolerance fraction defective p_1 : α = Probability of Type I error and β = Probability of Type II error then α and β are the maximum producer's and consumer's risks respectively. SPRT is defined as follows :

Let the result of the inspection of the ith unit be denoted by a Bernoulli variate X_i , *i.e.*,

$$X_i = 1, \text{ if } i\text{th item inspected is found to be defective}$$
$$= 0, \text{ otherwise.}$$

For the incoming lot quality 'p', if $f(x, p)$ represents the probability function of X then

$$f(1, p) = p \quad \text{and} \quad f(0, p) = 1 - p$$

Let p_{1m} and p_{0m} be the probabilities of getting d_m defectives in the sample ($X_1, X_2, \ldots, X_m$) of size m under H_1 and H_0 respectively. Then the Likelihood Ratio λ_m is given by :

$$\lambda_m = \frac{p_{1m}}{p_{om}} = \frac{\prod_{i=1}^{m} f(x_i, p_1)}{\prod_{i=1}^{m} f(x_i, p_0)} = \prod_{i=1}^{m} \frac{f(x_i, p_1)}{f(x_i, p_0)} = \frac{p_1^{dm} (1 - p_1)^{m - d_m}}{p_0^{dm} (1 - p_0)^{m - d_m}} \qquad \ldots(1{\cdot}23)$$

SPRT *is carried out as follows* : At each stage of the experiment, at the inspection of the mth for each possible integral value m, we compute λ_m and

(*i*) If $\lambda_m \ge A$, we terminate the process with rejection of the lot.
(*ii*) If $\lambda_m \le B$, we terminate the process with acceptance of the lot.
(*iii*) If $B < \lambda_m < A$, we continue the sampling by taking an additional observation, ...(1·23*a*)

where A and B are constants determined in terms of α and β and are given by

$$A = (1 - \beta)/\alpha \quad \text{and} \quad B = \beta/(1 - \alpha) \qquad \ldots(1{\cdot}23b)$$

For computational points of view, it would be much easier to deal with log λ_m rather than with λ_m. Thus SPRT can be restated as follows :

(*i*) If $\lambda_m \ge \log \log A$, reject the lot,
(*ii*) If $\log \lambda_m \le \log B$, accept the lot, and
(*iii*) If $\log B < \log \lambda_m < \log A$, continue sampling by taking one more observation. ...(1·23*c*)

$$\log \lambda_m = d_m \log\left(\frac{p_1}{p_0}\right) + (m - d_m)\log\left(\frac{1-p_1}{1-p_0}\right)$$

Hence accept the lot if

$$d_m \log\left(\frac{p_1}{p_0}\right) + (m - d_m)\log\left(\frac{1-p_1}{1-p_0}\right) \le \log B \;\Rightarrow d_m \le \frac{\log B - m\log\left(\frac{1-p_1}{1-p_0}\right)}{\log\left(\frac{p_1}{p_0}\right) - \log\left(\frac{1-p_1}{1-p_0}\right)} = a_m \text{ (say)} \quad \ldots(1\cdot24a)$$

Reject the lot if

$$d_m \log\left(\frac{p_1}{p_0}\right) + (m - d_m)\log\left(\frac{1-p_1}{1-p_0}\right) \ge \log A \Rightarrow d_m \ge \frac{\log A - m\log\left(\frac{1-p_1}{1-p_0}\right)}{\log\left(\frac{p_1}{p_0}\right) - \log\left(\frac{1-p_1}{1-p_0}\right)} = r_m \text{ (say)} \quad \ldots(1\cdot24b)$$

Continue sampling if $a_m < d_m < r_m$ ...(1·24c)

For each m, a_m and r_m are known as acceptance number and rejection number respectively.

Procedure. At each stage of the experiment, we compute a_m and r_m and we continue inspection as long as $a_m < d_m < r_m$. The first time when this inequality is violated, the inspection is stopped and then

(*i*) if $d_m \ge r_m$, lot is rejected, and

(*ii*) if $d_m \le a_m$, lot is accepted.

Remark. If we write

$$g_1 = \log(p_1/p_0),\; g_2 = \log\left(\frac{1-p_0}{1-p_1}\right);\; \log A = a,\; \log B = -b$$

and

$$s = \frac{\log(1-p_0/1-p_1)}{\log(p_1/p_0) - \log(1-p_1/1-p_0)} = \frac{g_2}{g_1+g_2} \quad \ldots(1\cdot24d)$$

then the acceptance and rejection lines L_1 and L_2 are given by the following equations :

Acceptance Line L_1 : $d_m = a_m = \dfrac{-b}{g_1+g_2} + \dfrac{mg_2}{g_1+g} \;\Rightarrow\; d_m = -h_1 + sm$ ...(1·24e)

where

$$h_1 = \frac{b}{g_1+g_2} \quad \ldots(1\cdot24f)$$

and $-h_1$ gives the intercept of the line L_1 on the d_m axis.

Rejection Line L_2 :

$$d_m = r_m = \frac{a}{g_1+g_2} + m\frac{g_2}{g_1+g_2} \;\Rightarrow\; d_m = h_2 + sm \quad \ldots(1\cdot24g)$$

where $h_2 = \dfrac{a}{g_1+g_2}$ ...(1·24h)

is the intercept of the line L_2 on the d_m axis.